COMMON FIXED POINT THEOREM OF FOUR MAPPINGS IN CONE METRIC SPACE

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Abstract

In this paper, we replace “max” by “co” in contractive condition, obtain a new common fixed point theorem of four mappings in cone metric space. The result unifies and extends many well-known fixed point theorems in metric spaces and cone metric spaces.

1. Introduction

In 1974, Ćirić [2] introduced and studied quasi-contraction mapping in metric space. The well known Ćirić’s result is that, see [2, 4, 8]:

**Theorem 1.** Let \((X, \rho)\) be a complete metric space. A mapping \(T : X \to X\) such that for some constant \(\lambda \in (0, 1)\) and for every \(x, y \in X\),

\[
\rho(Tx, Ty) \leq \lambda \max \{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \rho(x, Ty), \rho(y, Tx)\}.
\]

Then \(T\) possesses a unique fixed point.

Guang and Xian [5] replaced the real numbers by an ordered Banach space and defined a cone metric space. They proved some fixed point results in cone metric spaces.
Theorems for contractive mappings on cone metric spaces. Ilić and Rakočević [7] generalized Theorem 1 to cone metric space. They proved the following theorem.

**Theorem 2.** Let \((X, d)\) be a complete cone metric space, with normal cone. A mapping \(T : X \to X\) such that for some constant \(\lambda \in (0, 1)\) and for every \(x, y \in X\), there exists \(s \in C(T, x, y)\) such that

\[d(Tx, Ty) \leq \lambda s,\]

where \(C(T, x, y) = \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}\). Then \(T\) possesses a unique fixed point.

In [3], Das and Naik extended Theorem 1 to common fixed point of two maps. They proved the following theorem.

**Theorem 3.** Let \((X, \rho)\) be a complete metric space. Let \(f\) be a continuous self-map on \(X\) and \(g\) be any self-map on \(X\) that commutes with \(f\). Further, let \(f\) and \(g\) satisfy \(g(X) \subseteq f(X)\) and there exists a constant \(\lambda \in (0, 1)\) such that for every \(x, y \in X\),

\[\rho(gx, gy) \leq \lambda \max \{\rho(fx, fy), \rho(fx, gx), \rho(fy, gy), \rho(fx, gy), \rho(fy, gx)\}.\]

Then \(f\) and \(g\) have a unique common fixed point.

In [6], Ilić and Rakočević extended Theorem 3 to cone metric space. They proved the following theorem.

**Theorem 4.** Let \((X, d)\) be a complete cone metric space with normal cone. Let \(f, g : X \to X\) be two self-maps on \(X\) such that \(f\) is continuous and \(f\) and \(g\) are commuting and \(g(X) \subseteq f(X)\). Suppose that there exists a constant \(\lambda \in (0, 1)\) such that for every \(x, y \in X\), there exists \(s \in C(x, y)\) such that

\[d(gx, gy) \leq \lambda s,\]

where \(C(x, y) = \{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}\). Then \(f\) and \(g\) have a unique common fixed point.
In this paper, a new common fixed point theorem is given. The result generalizes corresponding results in metric space and cone metric space.

Now following [5], we give some definitions and auxiliary results.

Let $E$ be a real Banach space and $P$ be a cone in $E$, we define a partial ordering with respect to $P$ by $x \leq y$, if and only if $y - x \in P$. We shall write $x \ll y$, if $y - x \in \text{int } P$, where $\text{int } P$ denotes the interior of $P$.

**Definition 1** [5]. Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$, if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space.

Let $(X, d)$ be a cone metric space, $\{x_n\}$ be a sequence in $X$, and $x \in X$. If for every $c \in E$ with $0 \ll c$, there is $N$ such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to $x$, and $x$ is the limit of $\{x_n\}$.

Let $\{x_n\}$ be a sequence in $X$. If for any $c \in E$ with $0 \ll c$, there is $N$ such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in $X$. If every Cauchy sequence is convergent in $X$, then $X$ is called a complete cone metric space.

A mapping $T : X \to X$ is continuous, if $Tx_n \to Tx$ whenever $x_n \to x \in X$.

Let $E$ be a real linear space, $x \in E$, $A \subseteq E$, and $\lambda$ be a nonnegative real number. With $x \leq A$ and $x \leq \lambda A$, we denote there is an $s \in A$ such that $x \leq s$ and there is an $s \in A$ such that $x \leq \lambda s$, respectively.
Let $E$ be a real linear space, $A$ be a subset of $E$, with $\text{co}A$, we denote the convex hull of $A$.

## 2. Main Theorem

In this section, we shall prove our main theorem.

**Theorem 5.** Let $(X, d)$ be a complete cone metric space. Let $S, T, I, J$ be four mappings from $X$ to $X$, such that

(i) $S(X) \subseteq I(X)$ and $T(X) \subseteq J(X)$;

(ii) $I$ and $J$ are continuous;

(iii) $S, J$ and $T, I$ are commutative, i.e., $SJx = JSx$ and $TIx = ITx$ for all $x \in X$.

If $S, T, I, J$ satisfy following contractive condition:

\[
d(Sx, Ty) \leq \lambda \text{co}\{0, d(Jx, Iy), d(Jx, Sx), d(Iy, Ty), \frac{1}{2}(d(Jx, Ty) + d(Iy, Sx))\},
\]

(2.1)

for all $x, y \in X$. Then $S, T, I, J$ have a unique common fixed point in $X$.

**Proof.** Take an arbitrary $x_0 \in X$, let $y_0 = Jx_0$. Since $S(X) \subseteq I(X)$ and $T(X) \subseteq J(X)$, there are $x_1, x_2 \in X$ such that

\[
y_1 := Sx_0 = Ix_1, y_2 := Tx_1 = Jx_2.
\]

Generally, we have \{\{x_n\}\} and \{\{y_n\}\}, such that for all $n \in \mathbb{N}$

\[
y_{2n+1} := Sx_{2n} = Ix_{2n+1}, y_{2n+2} := Tx_{2n+1} = Jx_{2n+2}.
\]

From the contractive condition (2.1)

\[
d(y_1, y_2) = d(Sx_0, Tx_1)
\leq \lambda \text{co}\{0, d(Jx_0, Ix_1), d(Jx_0, Sx_0), d(Ix_1, Tx_1), \frac{1}{2}(d(Jx_0, Tx_1) + d(Ix_1, Sx_0))\}
\]

\[
= \lambda \text{co}\{0, d(y_0, y_1), d(y_0, y_1), d(y_1, y_2), \frac{1}{2}(d(y_0, y_2) + d(y_1, y_1))\}
\]
\[\Rightarrow d(y_1, y_2) \leq \lambda \text{co} \{0, d(y_0, y_1), d(y_0, y_1), d(y_1, y_2), d(y_0, y_1), d(y_1, y_2)\} \]
\[\Rightarrow d(y_1, y_2) \leq \lambda \text{co} \{0, d(y_0, y_1)\}.\]
\[d(y_2, y_3) = d(Sx_2, Tx_1)\]
\[\leq \lambda \text{co} \{0, d(Jx_2, Ix_1), d(Jx_2, Sx_2), d(Ix_1, Tx_1), \frac{1}{2} (d(Jx_2, Tx_1) + d(Ix_1, Sx_2))\}\]
\[= \lambda \text{co} \{0, d(y_2, y_1), d(y_2, y_3), d(y_1, y_2), d(y_1, y_2), d(y_2, y_3)\}\]
\[\Rightarrow d(y_2, y_3) \leq \lambda \text{co} \{0, d(y_0, y_1)\}\]
\[\Rightarrow d(y_2, y_3) \leq \lambda^2 \text{co} \{0, d(y_0, y_1)\}\]

Suppose that
\[d(y_{2n-1}, y_{2n}) \leq \lambda^{2n-1} \text{co} \{0, d(y_0, y_1)\},\]
and
\[d(y_{2n}, y_{2n+1}) \leq \lambda^{2n} \text{co} \{0, d(y_0, y_1)\}.\]

Then
\[d(y_{2n+1}, y_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})\]
\[\leq \lambda \text{co} \{0, d(Jx_{2n}, Ix_{2n+1}), d(Jx_{2n}, Sx_{2n}), d(Ix_{2n+1}, Tx_{2n+1})\},\]
\[\frac{1}{2} (d(Jx_{2n}, Tx_{2n+1}) + d(Ix_{2n+1}, Sx_{2n})))\]
\[= \lambda \text{co} \{0, d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2})\},\]
\[\frac{1}{2} (d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})))\]
\[
\Rightarrow \quad d(y_{2n+1}, y_{2n+2}) \leq \lambda c \{0, d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}^2), d(y_{2n+1}, y_{2n+1}^2)\}
\]

\[
\Rightarrow \quad d(y_{2n+1}, y_{2n+2}) \leq \lambda c \{0, d(y_{2n}, y_{2n+1})\}
\]

\[
\Rightarrow \quad d(y_{2n+1}, y_{2n+2}) \leq \lambda^{2n+1} c \{0, d(y_{2n}, y_{2n+1})\}
\]

\[
d(y_{2n+2}, y_{2n+3}) = d(Sx_{2n+2}, Tx_{2n+1})
\]

\[
\leq \lambda c \{0, d(Jx_{2n+2}, Ix_{2n+1}), d(Jx_{2n+2}, Sx_{2n+2}), d(Ix_{2n+1}, Tx_{2n+1}),
\]

\[
\frac{1}{2} (d(Jx_{2n+2}, Tx_{2n+1}) + d(Ix_{2n+1}, Sx_{2n+2}))
\]

\[
= \lambda c \{0, d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}),
\]

\[
\frac{1}{2} (d(y_{2n+2}, y_{2n+2}) + d(y_{2n+1}, y_{2n+3}))
\]

\[
\Rightarrow \quad d(y_{2n+2}, y_{2n+3}) \leq \lambda c \{0, d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+3}),
\]

\[
d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3})\}
\]

\[
\Rightarrow \quad d(y_{2n+2}, y_{2n+3}) \leq \lambda c \{0, d(y_{2n+1}, y_{2n+2})\}
\]

\[
\Rightarrow \quad d(y_{2n+2}, y_{2n+3}) \leq \lambda^{2n+2} c \{0, d(y_{2n}, y_{2n+1})\}
\]

Hence, it is proven by induction that for all \( n \)

\[
d(y_n, y_{n+1}) \leq \lambda^n c \{0, d(y_0, y_1)\}.
\]
There are $0 \leq a_n \leq 1$ such that
\[ d(y_n, y_{n+1}) \leq \lambda^n a_n d(y_0, y_1). \]
For $0 < m < n$,
\[ d(y_m, y_n) \leq d(y_m, y_{m+1}) + d(y_{m+1}, y_{m+2}) + \cdots + d(y_{n-1}, y_n) \]
\[ \leq (\lambda^m a_m + \cdots + \lambda^{n-1} a_{n-1}) d(y_0, y_1) \]
\[ \leq \frac{\lambda^m}{1-\lambda} d(y_0, y_1). \]
For $c \in E$, $0 << c$, there is $N$ such that $\frac{\lambda^m}{1-\lambda} d(y_0, y_1) << c$, for all $m \geq N$. So for $m \geq N$ and all $n$, we have
\[ d(y_m, y_n) << c. \]
Hence, $\{y_n\}$ is a Cauchy sequence. Since $X$ is complete, let $y_n \to y^*$ for some $y^* \in X$.

By the conditions (ii) and (iii)
\[ Jy^* = \lim_{n \to \infty} Jy_{2n+1} = \lim_{n \to \infty} JSx_{2n} = \lim_{n \to \infty} SJx_{2n} = \lim_{n \to \infty} Sy_{2n}, \]
\[ Iy^* = \lim_{n \to \infty} Iy_{2n} = \lim_{n \to \infty} ITx_{2n-1} = \lim_{n \to \infty} TIx_{2n-1} = \lim_{n \to \infty} Ty_{2n-1}. \]
By the contractive condition (2.1), for all $n$
\[ d(Sy_{2n}, Ty_{2n-1}) \]
\[ \leq \lambda \max \{0, d(Jy_{2n}, Iy_{2n-1}), d(Jy_{2n}, Sy_{2n}), d(Iy_{2n-1}, Ty_{2n-1}) \}, \]
\[ \frac{1}{2} (d(Jy_{2n}, Ty_{2n-1}) + d(Iy_{2n-1}, Sy_{2n})). \]
There are real $a_n, b_n, c_n, d_n$, with $a_n, b_n, c_n, d_n \geq 0$, $a_n + b_n + c_n + d_n \leq 1$, such that
\( d(S_{2n}, T_{2n-1}) \)
\[ \leq \lambda(a_n d(J_{2n}, I_{2n-1}) + b_n d(J_{2n}, S_{2n}) + c_n d(I_{2n-1}, T_{2n-1})) \]
\[ + d_n \frac{1}{2} (d(J_{2n}, T_{2n-1}) + d(I_{2n-1}, S_{2n})). \]
(2.2)

There are subsequences \( \{a_{n_m}\}, \{b_{n_m}\}, \{c_{n_m}\}, \{d_{n_m}\} \) of \( \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \) such that \( a_{n_m} \to a, b_{n_m} \to b, c_{n_m} \to c, d_{n_m} \to d \) as \( m \to \infty \) with \( a, b, c, d \geq 0, a + b + c + d \leq 1 \). Replace the subscript “\( n \)” by “\( n_m \)” in (2.2) and let \( m \to \infty \), we get
\[ d(J^*_y, I^*_y) \leq \lambda(a + d)d(J^*_y, I^*_y). \]
(2.3)

Since \( \lambda(a + d) < 1, d(J^*_y, J^*_y) = 0 \). Hence, \( I^*_y = J^*_y \).

Since
\[ d(S_{y*}, T_{2n-1}) \]
\[ \leq \lambda \inf \{0, d(J^*_y, I_{2n-1}), d(J^*_y, S_{2n}), d(I_{2n-1}, T_{2n-1}) \}, \]
\[ + \frac{1}{2} (d(J^*_y, T_{2n-1}) + d(I_{2n-1}, S_{2n})), \]
there are real \( a'_n, b'_n, c'_n, d'_n \geq 0, a'_n + b'_n + c'_n + d'_n \leq 1 \), such that
\[ d(S^*_{y}, T_{2n-1}) \]
\[ \leq \lambda(a'_n d(J^*_y, I_{2n-1}) + b'_n d(J^*_y, S_{y*}) + c'_n d(I_{2n-1}, T_{2n-1})) \]
\[ + d'_n \frac{1}{2} (d(J^*_y, T_{2n-1}) + d(I_{2n-1}, S_{y*})). \]
(2.4)

As the proof of (2.3), there are \( a', b', c', d' \geq 0, a' + b' + c' + d' \leq 1 \), such that
\[ d(S^*_{y}, I^*_y) \]
\[
\lambda(a'd(Jy^*, Iy^*) + b'd(Jy^*, Sy^*) + c'd(Iy^*, Iy^*)
+ d'\frac{1}{2}(d(Jy^*, Iy^*) + d(Iy^*, Sy^*)))
= \lambda(b' + \frac{1}{2}d')d(Sy^*, Jy^*).
\]

Since \(\lambda(b' + \frac{1}{2}d') < 1\), \(d(Sy^*, Jy^*) = 0\). Hence \(Sy^* = Jy^*\). Similarly, \(Ty^* = Iy^*\).

From
\[
d(Sy_{2n}, y_{2n+2}) = d(Sy_{2n}, Tx_{2n+1})
\leq \lambda\min\{0, d(Jy_{2n}, Ix_{2n+1}), d(Jy_{2n}, Sy_{2n}), d(Ix_{2n+1}, Tx_{2n+1})
+ \frac{1}{2}(d(Jy_{2n}, Tx_{2n+1}) + d(Ix_{2n+1}, Sy_{2n}))\}
\]

there are real \(a^*_n, b^*_n, c^*_n, d^*_n \geq 0\), \(a^*_n + b^*_n + c^*_n + d^*_n \leq 1\), such that
\[
d(Sy_{2n}, y_{2n+2})
\leq \lambda(a^*_n d(Jy_{2n}, Ix_{2n+1}) + b^*_n d(Jy_{2n}, Sy_{2n}) + c^*_n d(Ix_{2n+1}, Tx_{2n+1})
+ d^*_n \frac{1}{2}(d(Jy_{2n}, Tx_{2n+1}) + d(Ix_{2n+1}, Sy_{2n}))). \tag{2.5}
\]

As the proof of (2.3), there are \(a^*, b^*, c^*, d^* \geq 0\), \(a^* + b^* + c^* + d^* \leq 1\), such that
\[
d(Jy^*, y^*) \leq \lambda(a^* + d^*)d(Iy^*, y^*).
\]

Since \(\lambda(a^* + d^*) < 1\), \(d(Jy^*, y^*) = 0\). So \(Jy^* = y^*\). Therefore, \(y^*\) is a common fixed point of \(S, T, I, J\).

Now, if \(x^*\) is also a common fixed point of \(S, T, I, J\), then
\[
d(x^*, y^*) = d(Sx^*, Ty^*)
\]
This implies that $x^* = y^*$. Hence, the common fixed point of $S, T, I, J$ is unique.

Remark 1. In Theorem 5, let $I = J = id$ (the identity operator on $X$), or $S = T$, or $X$ be a metric space, we can get many versions of fixed point and common fixed point theorems.

References